COHERENT PREFERENCES*

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ABSTRACT

The term coherence has been used to describe probability assignments and also actions of a decision maker. Here we describe a related concept of coherent preferences. Willingness to accept one side or either side of a bet are examples of preferences. A set of preferences is called (preference-reversal) incoherent if reversal of some subset results in a uniform increase in utility, as in the case of a sure win for a collection of bets. In both probability and statistical models (where preferences are conditional on data) we use "separating hyperplane" theorems to show that coherence implies existence of a probability measure from which the stated preferences could have been inferred. In one statistical model involving fairly general spaces, coherence implies existence of a finitely additive measure having this property.

1. Introduction

The term coherence has been used in probability and statistics in a number of senses. In a probability model where A denotes an event and x is the indicator function of A the payoff of a bet on A is $k(x-p)$ where kp is the stake and p is a stated probability of A used in reckoning the betting odds. For a class of events A_i and corresponding values p_i, not necessarily obeying standard probability axioms, the values p_i (or the corresponding odds) are called coherent (or fair or consistent) if no stakes exist which guarantee a sure win. For certain sets of events, coherence of the "probabilities" p_i (or corresponding odds) implies that the p_i values must satisfy the usual probability axioms (deFinetti (1937; 1949; 1964; 1972, Section 5.9), Shimony (1955), Kemeny (1955), Heath and Sudderth (1972)).

Coherence has more recently been used in connection with statistical procedures. Cornfield (1969) refers to coherence of probabilities assigned to states of nature θ after observing data x, that is posterior probabilities, or values playing the role of posterior probabilities. Here coherence of such values means that there are no stakes which give a positive expectation for every θ. This might perhaps be called a sure win, but in fact it is a weaker guarantee than in the probability model because the expectation over x involved here guarantees a win in the long run, not on any individual trial. It was shown by Cornfield and independently by Freedman and Purves (1969) that in finite models coherence of posterior "probabilities" assigned to all possible outcomes implies their agreement with values calculated by Bayes theorem for some prior distribution.

Similar results of Quiring (1972) pertain to cases where probabilities are assigned only to some subsets of θ values. For other extensions of the theory see Dawid and Stone (1972, 1973).

Lindley (1971, p. 6) speaks of coherent actions:

> A decision maker whose actions agree with these axioms has variously been described as <u>rational</u>, <u>consistent</u>, or <u>coherent</u>. We shall use the last term because it effectively captures the idea that the basic principle behind the axioms is that our judgements should fit together or cohere.

Summing up, we have cited examples of the study of coherent probability values and coherent odds both in probability models (deFinetti, Shimony, Kemeny, Heath and Sudderth) and in "statistical inference" models (Cornfield, Freedman and Purves, Quiring), and of coherent action or behavior, again in a statistical model (Lindley).

The present report uses the term coherent in mathematical senses which are in much the same in spirit but different in emphasis. Instead of referring to probabilities or to odds or to actions, we speak here of coherent <u>preferences</u>. Both probability and statistical models are discussed, and results are obtained which are analogous to those mentioned above.

A simple example may be helpful. Let B denote the event that horse Beetlebaum wins a certain race. From the extreme Bayesian point of view any person can attach a unique subjective value to P(B) whether or not he knows anything about horse racing in general or Beetlebaum in particular. There exist less extreme theories (Smith (1961), Dempster (1968)) involving upper and lower probabilities. Suppose our subject, Peter, believes $p = P(B)$ lies in the range $1/4 < p < 3/4$, but don't know where. Let (α, β) denote a bet which pays β if B occurs and α otherwise. Peter would presumably accept bets (4, -1), (-1, 4) but would reject bets (2, -1), (-1, 2), since the former have positive expectation for all $1/4 < p < 3/4$ but the latter do not. From our present point of view Peter would not be faulted for any of the individual choices mentioned above. But if the bets (2, -1), (-1, 2) were offered simultaneously we would say his preferences for (0, 0) (that is, no bet) over (2, -1) and for (0, 0) over (-1, 2) are incoherent because by reversing both choices he has a sure win of +1 no matter who wins the race.

As defined in this report, ("preference reversal" or PR) incoherence of a set of preferences means existence of a subset whose simultaneous reversal guarantees a higher utility for every outcome. For a number of probability models it is shown that coherence of preferences is equivalent to an ordering consistent with expected values calculated from some probability measure. In the statistical model similar results imply agreement with Bayesian preferences for some prior distribution.

2. Relationships between bets and preferences

Let $(\alpha, -\beta)$ denote a prospect which gives α if A occurs and $-\beta$ if the complement $\bar{A}$ occurs, and let c be a positive scalar. The preference $c(\alpha, -\beta) \succ (0,0)$ is equivalent to willingness of the subject, Peter, to bet on A (accept Paul's bet against A) with odds corresponding to the probability $P(A) = \beta/(\alpha + \beta)$. Peter's willingness to bet either for or against A would be represented by the pairs of preferences $(\alpha,-\beta) > (0,0)$ and $(-\alpha,\beta) > (0,0)$. This pair might be replaced indifference statements: Peter is indifferent between $(\alpha,-\beta)$ and $(0,0)$, denoted by $(\alpha,-\beta) \approx (0,0)$, or equivalently by $(\alpha,-\beta) \approx (-\alpha,\beta)$. Our intuition may be aided by thinking of this indifference as corresponding to $P(A) = \beta/(\alpha + \beta)$ and the preference $(\alpha,-\beta) > (0,0)$ as corresponding to $P(A) \geq \beta/(\alpha + \beta)$.

In finite spaces coherence problems can be thought of geometrically. Any bet or preference limits probability vectors to a subset of a simplex. A combination of bets or preferences is incoherent if the intersection of the corresponding subsets is empty.

Now consider a case having three exclusive and exhaustive outcomes A_1, A_2, A_3. Let us say a simple bet is one which wins α_i if A_i occurs and loses β_i if $\bar{A}_i$ occurs. We ask whether a preference such as $(\alpha_1, \alpha_2, -\beta) > (0,0,0)$ is equivalent to willingness to accept some combination of simple bets. Since the stated preference corresponds to a probability vector (p_1, p_2, p_3) satisfying $\alpha_1 p_1 + \alpha_2 p_2 - \beta p_3 \geq 0$ and since the hyperplane boundary in p-space cannot in general be duplicated by intersection of regions based on simple bets, we conclude that the present models involving preferences are more general than models based on simple bets. For this reason the (finite space) results given in Section 5 would not follow either from those of Freedman and Purves (1969) or Cornfield (1969) or Quiring (1972, Chapter II).

The betting viewpoint could indeed be extended to include "compound" bets having three or more unequal payoffs. However, the conceptual formulation in terms of preferences rather than bets seems preferable for the present report.

3. Mathematical Preliminaries

Our main tools are familiar "separating hyperplane" theorems of linear algebra and their extensions. This type of argument has previously been used on similar problems by Freedman and Purves (1969), Heath and Sudderth (1972), Quiring (1972), Dawid and Stone (1972), and Pierce (1973). We state three lemmas for future reference.

Definition 1. For any vector $w = (w_1,\ldots,w_r)'$ we will write

$w \geqq 0$ if w is non-negative, that is $w_i \geq 0$ all i,

$w > 0$ if w is positive, that is $w_i > 0$ all i,

$w \geq 0$ if w is semi-positive, that is $w \geqq 0$ and $w \neq 0$.

The first lemma is given for example by Gale (1960), Theorem 2.10. The bracketed part is a variation used only incidentally in the present report.

Lemma 1. Let M be an m-by-n matrix. Exactly one of the following alternatives holds: (i) there exists an n-by-1 vector $v \geq 0$ such that $Mv > 0$ [$Mv \geq 0$], or (ii) there exists an m-by-1 vector $w \geq 0$ [$w > 0$] such that $-w'M \geqq 0$.

In applications of Lemma 1, the columns of M are utility vectors (called prospects); w is a probability vector in R^m corresponding to a sample space of m outcomes; v gives a weighted combination of the n columns of M. Let $s = 1,\ldots,m$ label the rows and $t = 1,\ldots,n$ the columns. In generalizations either s or t take infinitely many values. Lemma 2 is a variant of the Lemma of the same number in Pierce (1973). Our proof is more elementary in not requiring the theory of duality of L_p spaces.

Lemma 2. Let (T, B, λ) be a σ-finite measure space and let L_1 and

L_∞ denote respectively λ-integrable and bounded functions on T. If $M_s(t) \in L_1$ for $s = 1,\ldots,m$, then exactly one of the following alternatives holds: (i) there exists $v(t) \in L_\infty$, $v(t) \geq 0$ for all t, such that

$$(3.1) \qquad \int v(t)\, M_s(t)\, d\lambda > 0 \quad \text{for all} \quad s = 1,\ldots,m,$$

or (ii) there exists a vector $w = (w_1,\ldots,w_m)' \geq 0$ such that $\lambda\{t \mid \sum w_s M_s(t) > 0\} = 0$.

<u>Proof</u>. We define a convex cone in m-space by $C = \{(a_1,\ldots,a_m) \mid a_s = \int v(t)\, M_s(t)\, d\lambda,\ v(t) \geq 0,\ v(t) \in L_\infty\}$. Let Q denote the positive orthant of vectors ≥ 0. If (i) holds, then $C \cap Q \neq \phi$. Assume (i) is false so that $C \cap Q = \phi$. Then there exists a separating hyperplane, whose normal direction we call w, such that $w'q \geq 0$ for all $q \in Q$ (implying $w \in Q$) and $w'c \leq 0$ for all $c \in C$. The last inequality is equivalent to

$$(3.2) \qquad \int v(t) \sum w_s M_s(t)\, d\lambda \leq 0 \quad \text{for all} \quad v(t) \in L_\infty,\ v(t) \geq 0.$$

Define $S_w^\epsilon = \{t \mid \sum w_s M_s(t) > \epsilon\}$, $S_w^0 = \lim_{\epsilon \to 0} S_\pi^\epsilon$, and $v_\epsilon(t)$ = indicator function of S_w^ϵ. Then

$$(3.3) \qquad 0 \geq \int v_\epsilon(t) \sum w_s M_s(t)\, d\lambda \geq \epsilon \cdot \lambda(S_w^\epsilon),$$

so that $\lambda(S_w^\epsilon) = 0$ and $\lambda(S_w^0) = \lim \lambda(S_w^\epsilon) = 0$. Thus when (i) is false

(ii) is true, and it is straightforward to show that (i) and (ii) cannot both be true.

In Lemma 3 both s and t range over arbitrary spaces. Lemma 3 is a restatement in the present notation of Theorem 1′ of Heath and Sudderth (1973). (Theorem 1′ is a strengthening of Theorem 1 of Heath and Sudderth (1972) in which "$\sum_{i=1}^{n} c_i f_{t_i}(s) > 0$ for all $s \in S$" is replaced by "$\inf_{s\in S} \sum_{i=1}^{n} c_i f_{t_i}(s) > 0$" and "or both" is replaced by "but not both."

Lemma 3. Let S and T be sets and let $\{M_t(s): t \in T\}$ be a family of bounded, real-valued functions defined on S. Exactly one of the following alternatives holds: (i) there exist $t_1,\dots,t_n \in T$ and $v_1 \geq 0,\dots,v_n \geq 0$ such that

$$(3.4) \qquad \inf_{s\in S} \sum_{i=1}^{n} v_j M_{t_j}(s) > 0,$$

or (ii) there exists a finitely additive probability P on S such that

$$(3.5) \qquad E_p M_t = \int M_t(s)\, dP(s) \leq 0 \quad \text{for all } t \in T.$$

4. Probability Models

We begin with a space S of points s to be thought of as a space of simple (or elementary) outcomes. For the moment we avoid the assumption of any probability measure.

Definition 2. A prospect $g(s)$ is a bounded real valued function defined on S. It is to be considered as reward of $g(s)$ given to our subject, Peter, when outcome s is observed. In the subsequent calculations it will be seen that values of g are handled like utilities, but we do not require an explicit assumption of this in our treatment.

Definition 3. $g' > g$ means Peter prefers g' to g, that is, given a choice of the two, he would take g' (the case of indifference is considered to be included; the choice of g' being consistent with indifference).

In usual utility theoretic frameworks $g' > g$ would be equivalent to $h = g' - g > 0$, that is, a reward of $g'(s) - g(s)$ if s occurs is preferred to no reward at all. Further $h > 0$ would be equivalent to $\lambda h > 0$ where λ is any positive constant. For $0 < \lambda < 1$ this is of course like saying $g' > g$ if and only if any mixed prospect of the form "g' with probability λ and g with probability $1 - \lambda$" is preferred to g. These assertions are not given the status of axioms here but are included to motivate definitions of coherence.

4.1. Finite Models

Definition 4. Let $g_t' > g_t$ be a set of preferences for $t = 1,\ldots,n$. For a finite space $S = \{1,\ldots,m\}$ this set is called preference-reversal- (or PR-) incoherent if there exist $v_1 \geq 0,\ldots,v_n \geq 0$ such that

$$(4.1) \qquad \sum_{t=1}^{n} v_t(g_t(s) - g_t'(s)) > 0 \quad \text{for } s = 1,\ldots,m.$$

Here and in the sequel preferences which are not PR-incoherent are called PR-coherent.

It is clear that the t values for which $v_t \neq 0$ single out a set of preferences such that a weighted combination violates the stated preferences uniformly in the outcomes s.

Definition 5. A set of preferences $g'_t > g_t$, $t = 1,\ldots,n$, is called w-coherent if there exists a probability vector $w_1,\ldots,w_m$ such that

$$(4.2) \qquad \sum_{s=1}^{m} w_s g'_t(s) \geq \sum_{s=1}^{m} w_s g_t(s) \quad \text{for} \quad t = 1,\ldots,n.$$

Theorem 1. Preferences are PR-coherent if and only if they are w-coherent for some w.

Proof. Apply Lemma 1 with $M_{st} = g_t(s) - g'_t(s)$.

Since Definitions 4 and 5 are equivalent we can drop the prefixes PR and w and speak simply of coherent preferences.

Example 1. Let $g'_t - g_t = h_t$, $h_1 = (1, -2, 0)$, $h_2 = (-2, 1, 0)$. Then $-h_1 - h_2 = (1, 1, 0)$ so that a reversal of preferences improves Peter's lot for $s = 1,2$ and leaves it unchanged for $s = 3$. The improvement is not uniform in s, and the preferences are in fact coherent. A unique w gives a consistent ordering: $w = (0, 0, 1)'$. It would be possible to formulate an alternative theory in which the above h_1 and h_2 might be called weakly incoherent, the negation, say strong coherence, would correspond to the existence of $w > 0$ (i.e., $w_s > 0$, all s) giving a consistent ordering. The bracketed version of Lemma 1 would be relevant to this formulation. The distinction is essentially the same as that between "fair" and "strictly fair" in the sense of Shimony (1955) and Kemeny (1955), and between "strict" and "weak" coherence of Quiring (1972, Chapter II). DeFinetti (1972, p. 91) favors the (weak) coherence of our definitions 4 and 5.

4.2. Infinite Models

Suppose $s = 1,\dots,m$, $t \in T$, where (T, B, λ) is a measure space. Then $h_t(s) = g_t'(s) - g_t(s) \succ 0$ represents an infinity of preferences involving a finite number of alternatives. PR-incoherence can be defined as the existence of a bounded λ-measurable function $v(t) \geq 0$ such that $\int v(t)h_t(s)\, d\lambda < 0$ for all s, and w-coherence means existence of a probability vector w such that $\sum w_s h_t(s) \geq 0$ a.e. (λ). Lemma 2 shows that PR-coherence is equivalent to w-coherence for some w.

Finally, we may let s and t range over arbitrary spaces S and T. Define preferences $h_t(s) = g_t'(s) - g_t(s) \succ 0$, $t \in T$, to be PR-incoherent if there exist $t_1,\dots,t_n \in T$, $v_1 \geq 0,\dots,v_n \geq 0$, such that

$$(4.3) \qquad \sup_{s \in S} \sum_{i=1}^{n} v_i h_{t_i}(s) < 0,$$

and define them to be w-coherent if there exists a finitely additive probability measure w such that $\int h_t(s)\, d\, w \geq 0$ for all $t \in T$. Then Lemma 3 implies equivalence of PR-coherence and w-coherence for some w.

<u>Example 2</u>. If $w = (1/2, 1/4, 1/8, \dots)$, $h_1 = (-1, +3, 0, \dots)$, $h_2 = (0, -1, +3, 0, \dots)$, etc., then $Eh_j > 0$, so that each h is strictly preferred to 0. Nevertheless $h_1 + 4h_2 + 4^2h_3 + \dots$ is strictly negative. This shows that an infinite combination of fair bets can be unfair (at least when stakes are unbounded). Thus any theory of

coherence would seem to need either bounded stakes or finite combinations.

Example 3. Let $S = T = \{1, 2, \ldots\}$, $h_1(1) = 1$, $h_t(1) = 0$ for $t = 2, 3, \ldots$ The preferences are coherent because no positive combination gives a negative reward when $s = 1$. A probability vector consistent with these preferences is $(1, 0, 0, \ldots)$.

Example 4. $h_1 = (-1, 2, 0, \ldots)$, $h_2 = (0, -1, 2, 0, \ldots)$, etc. For a combination $c_1h_1 + c_2h_2 + \ldots$ to be strictly negative we would need $c_1 > 0$, $-2c_1 + c_2 > 0$, $-2c_2 + c_3 > 0$, etc. No finite number of strictly positive c_j will suffice so that the preferences are PR-coherent. Any probability vector with $p_{j+1} \geq \frac{1}{2} p_j$ gives preferences which agree.

Example 5. $h_1 = (-2, 1, 0, \ldots)$, $h_2 = (0, -2, 1, 0, \ldots)$, etc. The same argument shows the preferences $h_s > 0$ are coherent. Here however the probability measure must attach zero probability to each individual outcome and so be only finitely, not countably additive.

5. Statistical Models

In the statistical model the set S of outcomes s is replaced by a set Θ of parameter values θ. A new ingredient is the sample space $\mathcal{X}$ having points x. In the present report we restrict $\mathcal{X}$ to be countable. The function $p(x;\theta)$ denotes the likelihood, that is a probability law: $P(X = x|\theta) = p(x;\theta)$; $\sum_x p(x;\theta) = 1$ for each θ. A prospect now is a reward to Peter of $g(\theta)$ when θ is the true parameter value. Preferences are made conditional on observed data x; that is, Peter expresses his preference for g or g' only after observing x. We write $h(x,\theta)$ for $g'(\theta) - g(\theta)$ when $g' > g$ given x. A set of preferences then indexed by $t \in T$ is a set of functions $h_t(\theta)$ depending on x and θ according to

$$(5.1) \qquad h_t(x,\theta) = \begin{cases} g_t'(\theta) - g_t(\theta) & \text{when } x = x_t \\ 0 & \text{otherwise.} \end{cases}$$

For fixed θ the expectation of $h_t(x,\theta)$ with respect to x is

$$(5.2) \qquad f_t(\theta) = h_t(x_t,\theta)\; p(x_t,\theta) \;.$$

<u>5.1. Finite T and Θ</u>

Suppose $\theta = 1,\ldots,m$, $t = 1,\ldots,n$.

<u>Definition 6</u>. The set of preferences of the form (5.1) for $t = 1,\ldots,n$ is called PR-incoherent if there exist $v_1 \geq 0,\ldots,v_n \geq 0$ such that

$$(5.3) \qquad E_\theta \textstyle\sum_t v_t h_t\,(x,\theta) = \sum_t v_t f_t(\theta) < 0 \quad \text{for} \quad \theta = 1,\ldots,m.$$

The interpretation is that preference reversal, with suitable weights, would increase the <u>expected</u> reward to Peter for every parameter value.

For prior probability vector $w = (w_1,\ldots,w_m)'$, denote expectation with respect to w by

$$(5.4) \qquad E_w\, \varphi(\theta) = \sum_{\theta=1}^{m} w_\theta\, \varphi(\theta) \;.$$

Then the marginal distribution of x is

$$(5.5) \qquad p_w(x) = E_w\, p(x,\theta) = \sum w_\theta\, p(x,\theta),$$

the posterior density is

$$(5.6) \qquad w(\theta|x) = w_\theta\, p(x,\theta)/p_w(x) \;,$$

and the posterior expectation of $h_t(x,\theta)$ in (5.1) is

$$(5.7) \qquad E_{w|x_t}\, h_t(x,\theta) = (p_w(x_t))^{-1} \;\Sigma_\theta\; w_\theta\, p(x_t,\theta)\, h_t(x_t,\theta)$$

$$= (p_w(x_t))^{-1}\, E_w f_t(\theta).$$

We require a simple convention to cover the case where the denominator $p_w(x)$ in (5.6) and (5.7) is zero. If for some preference $h_t(x,\theta) > 0$ where $h_t(x,\theta)$ is given by (5.1), $p(x,\theta)$ and w_θ are such that $p_w(x_t) = 0$, then since the preference in question has probability zero of being exercised, we define the posterior expectation of $h_t(x,\theta)$ by

$$(5.8) \qquad E_{w|x_t}\, h_t(x,\theta) = \begin{cases} (p_w(x_t))^{-1}\, E_w f_t(\theta) & \text{if } p_w(x_t) \neq 0 \\ 0 & \text{if } p_w(x_t) = 0 \end{cases}$$

<u>Definition 7</u>. In the statistical model a set of preferences $h_t(x,\theta)$, $t = 1,\ldots,n$, is called w-coherent if there exists a (prior) probability vector w such that

$$(5.9) \qquad E_{w|x_t}\, h_t(x,\theta) \geq 0 \quad \text{for} \quad t = 1,\ldots,n.$$

Theorem 2. A set of preferences (5.1) for $t = 1,\ldots,n$ is PR-coherent if and only if it is w-coherent for some w.

Proof. Assume w-coherence for some w. Define $U(x) = \{\theta | p(x,\theta) > 0\}$. Then $p_w(x_t) = 0$ implies $U(x_t) = U_t$, say has prior (or w) probability zero. Note that

$$(5.10) \qquad E_w f_t(\theta) = \{\sum_{\theta \in U_t} + \sum_{\theta \notin U_t}\} h_t(x_t,\theta)\ p(x_t,\theta)\ w_\theta .$$

If $p_w(x_t) \neq 0$ we have by (5.8) and (5.9), $E_w f_t(\theta) \geq 0$. If $p_w(x_t) = 0$ then the first summation in (5.10) is zero because U_t has prior probability zero and the second is zero because $p(x_t,\theta) = 0$. Therefore $E_w f_t(\theta) = 0$ in this case, and we can apply Lemma 1 with $M_{st} = -f_t(s)$. Countrariwise, PR-coherence, or nonexistence of the v vector implies existence of w such that $E_w f_t(\theta) \geq 0$ for all t. By (5.8) this implies $E_{w|x_t} h_t(x,\theta) \geq 0$ for all t, that is, w-coherence.

We note some special cases.

If all x_t were the same, say $x_t = x_0$ for all t, then one could alternatively appeal to Theorem 1 to argue that PR-coherence implies existence of a conditional probability, given x_0, consistent with the preferences. Dividing this by the likelihood $p(x_0,\theta)$ and normalizing gives a prior measure known to exist by Theorem 2. We emphasize that in Theorem 1 we may have multiple preferences for some x values while for others there may be none at all.

The case studied by Cornfield (1969) and Freedman and Purves (1969), wherein Peter states odds for every subset of θ values given every x, corresponds to preferences $h_t(x_t,\theta) \succ 0$ and $-h_t(x_t,\theta) \succ 0$ where for each t, $h_t(x_t,\theta)$ takes only two values (whose ratio is determined by the odds) as a function of θ. The above authors show that this large set of preferences makes the prior measure unique.

5.2. Some cases where T and Θ are not both finite

In extending the model of Section 5.1 we continue to restrict $\mathfrak{X}$ to be a countable space to avoid difficulties in defining conditional distributions.

Lemma 2 is relevant to the case where Θ remains finite but T does not. Rather than take (T, B, λ) to be an arbitrary measure space for simplicity, we take $T = \{1, 2, \ldots\}$ and λ = counting measure. Then Lemma 2 asserts that either there exist $w_1 \geq 0,\ldots,w_m \geq 0$ such that

$$(5.11) \qquad \sum_{s=1}^{m} w_s M_s(t) \leq 0 \quad \text{for} \quad t = 1,2,\ldots,$$

or there exist $v_1 \geq 0$, $v_2 \geq 0,\ldots,$ such that

$$(5.12) \qquad \sum_{t=1}^{\infty} v_t M_s(t) > 0 \quad \text{for} \quad s = 1,\ldots,m.$$

Definitions 6 and 7 can be extended by taking $n = \infty$, and Theorem 2 continues to hold by the same argument.

Finally Lemma 3 applies to the case where the space Θ is arbitrary. Let w be a finitely additive measure on Θ and let E_w denote expectation with respect to w. We define the posterior measure on Θ by

$$(5.13) \qquad w(d\theta|x) = (p(x,\theta)/p_w(x))\; w(d\theta) \quad \text{where} \quad p_w(x) = E_w p(x,\theta).$$

and denote posterior expectation by $E_{w|x}$. Definition 7 is changed to allow w to be finitely additive and (5.9) holds for all t in an arbitrary set T. Definition 6 of PR-incoherence is altered to read: there exist $t_1,\ldots,t_n \in T$, $v_1 \geq 0,\ldots,v_n \geq 0$ such that

$$(5.14) \qquad \sup_{\theta\in\Theta} \sum_{i=1}^{n} v_i f_{t_i}(\theta) < 0$$

(the definition (5.2) of $f_t(\theta)$ is still used). Lemma 3 then shows that PR-coherence is equivalent to w-coherence for some finitely additive w.

Example 6. $\Theta = \{0, \pm 1, \pm 2, \ldots\}$, $p(x;\theta) = 1/2$ if $x = \theta \pm 1$ and $= 0$ otherwise. For a given x_o the structural probability distribution has mass 1/2 at each of the two points $x_o \pm 1$ (Fraser, 1971). Therefore when x_o is observed the structural probabilist is indifferent between $h_{x_o} =$ $(\ldots 0, +1, 0, -1, 0, \ldots)$ (where the middle zero is in the x_o position) and the zero vector. Are these preferences coherent in the present technical sense? Yes they are because any finite combination of necessity only involves a finite number of θ-values, and so the expectation of any finite combination cannot be positive for infinitely many θ-values. We conclude that the preferences agree with an ordering by a finitely additive prior distribution, and in fact any one which attaches probability zero to each θ value will do. The betting scheme proposed by Buehler (1971) (related to Blackwell's (1951) work on admissiblity, as pointed out by Stone (1972)) corresponds to taking the finite linear combination $H = -h_1 - h_2 - \ldots$ for which $EH = -1/2$ for $\theta = 0$ or 1 and $EH = 0$ otherwise. This H fails for two reasons to demonstrate incoherence: (i) The linear combination is infinite rather than finite, and (ii) $\sup_\theta EH = 0$ rather than < 0. The remarks on "strong" and "weak" coherence made in Example 1, Section 4.1, are relevant here; the present example is seen to demonstrate a kind of "weak" incoherence.

5.3. Replacing the probability law of x by preferences

Let Θ and $\mathfrak{X}$ be finite spaces. We now replace the probability law $p(x,\theta)$ by a set of preferences conditional on θ. For example $g_o'(x) > g_o(x)$ could mean that given that $\theta = \theta_o$ Peter prefers a reward of $g_o'(x)$ to $g_o(x)$. A given probability function $p(\cdot, \theta_o)$ would of course determine all such (conditional on θ_o) preferences. A set of conditional preferences could be incoherent, or could determine a unique conditional probability law, or could be consistent with a number of probability laws. The preference $g_o' > g_o$ can be written $h > 0$ where

$$(5.15) \qquad h = h(x,\theta) = \begin{cases} g_o'(x) - g_o(x) & \text{if } \theta = \theta_o \\ 0 & \text{otherwise} \end{cases}$$

Certain beliefs concerning the conditional probability law of x given θ can be represented by a set of preferences $h_t > 0$, $t \in T_1$, where each $h_t = h_t(x,\theta)$ is nonzero only when θ_t.

Similarly as in the previous sections Peter can have conditional preferences given particular x values, for example, $k_o'(\theta) > k_o(\theta)$ given $x = x_o$. Putting

$$(5.16) \qquad h = h(x,\theta) = \begin{cases} k_o'(\theta) - k(\theta) & \text{if } x = x_o \\ 0 & \text{otherwise} \end{cases}$$

a set of preferences conditional of x is represented by $h_t > 0$, $t \in T_2$, where each h_t is nonzero only when $x = x_t$.

Definition 8. The conditional preferences $\{h_t, t \in T_1 \cup T_2\}$ are called PR-incoherent if there exists $v_1 \geq 0, \ldots, v_n \geq 0$, $t_1, \ldots, t_n \in T_1 \cup T_2$ such that

$$(5.17) \qquad \sum v_i h_{t_i} (x,\theta) < 0 \quad \text{for all } x, \theta.$$

We remark that if there exist (x_o, θ_o) such that $\theta_t \neq \theta_o$ for all $t \in T_1$, $x_t \neq x_o$ for all $t \in T_2$, then (5.17) fails for $(x,\theta) = (x_o, \theta_o)$ so that the preferences are necessarily PR-coherent.

Definition 9. Sets T_1 and T_2 of preferences $h_t > 0$ conditional on θ and x respectively are called π-p-coherent if there exists a prior measure $\pi(\theta)$ and a likelihood $p(x,\theta)$ such that

$$(5.18) \qquad \begin{aligned} &E_{x|\theta}\, h_t(x,\theta) \geq 0 \qquad \text{for all } t \in T_1, \text{ and} \\ &E_{\theta|x}\, h_t(x,\theta) \geq 0 \qquad \text{for all } t \in T_2. \end{aligned}$$

Zero values of π and p are not precluded, and in (5.18) the following conventions are to be understood: when $\pi(\theta_o) = 0$ and $\theta_t = \theta_o$ then $E_{x|\theta}\, h_t(x,\theta) = 0$; when $\sum_\theta \pi(\theta)\, p(x_o,\theta) = 0$ and $x_t = x_o$ then $E_{\theta|x}\, h_t(x,\theta) = 0$.

Theorem 3. Conditional preferences are PR-coherent if and only if they are π-p-coherent for some prior $\pi(\theta)$ and some likelihood $p(x,\theta)$.

Proof. If the preferences are PR-coherent then Lemma 1 implies the existence of $w(x,\theta) \geq 0$ (not identically zero) such that

$$(5.19) \qquad \sum_x \sum_\theta \; w(x,\theta)\, h_t(x,\theta) \ge 0 \quad \text{for all} \quad t \in T_1 \cup T_2.$$

We may assume w to be normalized to a probability measure on $\mathfrak{X} \times \Theta$, and define $\pi(\theta) = \sum_x w(x,\theta)$, $p(x,\theta) = w(x,\theta)/\pi(\theta)$ when the $\pi(\theta) \neq 0$, and $p(x,\theta)$ is arbitrary when $\pi(\theta) = 0$. For $t \in T_1$, $E_{x|\theta}\, h_t(x,\theta) = 0$ unless $\theta = \theta_t$. When $\theta = \theta_t$ and $\pi(\theta_t) = 0$, then $E_{x|\theta}\, h_t(x,\theta) = 0$ be convention. When $\theta = \theta_t$ and $\pi(\theta_t) \neq 0$ then

$$\begin{aligned} E_{x|\theta}\, h_t(x,\theta) &= (\pi(\theta_t))^{-1} \textstyle\sum_x h_t(x,\theta)\, w(x,\theta) \\ &= (\pi(\theta_t))^{-1} \textstyle\sum_x \sum_\theta h_t(x,\theta)\, w(x,\theta) \\ &\ge 0, \text{ by (5.19).} \end{aligned}$$

The proof for $t \in T_2$ is similar, as is the converse.

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